

Microscopic formulation of the S-matrix in AdS/CFT

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ABSTRACT: We develop the derivation we proposed in [37] of the dressing phase of the S-matrix in the AdS/CFT correspondence in the framework of the underlying bare integrable model. We elaborate the configuration of the Bethe roots describing the physical vacuum, which consists of a long Bethe string stretched along the imaginary axis and stacks distributed along the real axis. We determine the distribution of all Bethe roots in the thermodynamic limit. We then directly compute the scattering phase of the fundamental excitations over the physical vacuum and reproduce the BHL/BES dressing phase.

KEYWORDS: AdS-CFT Correspondence, Bethe Ansatz, Exact S-Matrix.

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1. Introduction

Integrability has been providing us with new insights into the duality between the $\mathcal{N} = 4$ super Yang-Mills theory and the superstring theory in $AdS_5 \times S^5$. After the discovery of integrability in the one-loop super Yang-Mills theory [1, 2] and in the classical superstring theory [3], a lot of progress has been made toward the all-order/quantum integrability in the full theory of the planar AdS/CFT correspondence. As a monumental result, there has emerged a novel integrable model [4, 5] which is expected to describe the spectrum of the infinitely long Yang-Mills operators as well as that of the infinitely long quantum strings [6–8], at arbitrary values of the 't Hooft coupling constant λ .

The integrable model is characterized by the dispersion relation [9] and the S-matrix [5, 10, 11] of the fundamental particles. The system exhibits the centrally extended $\mathfrak{psu}(2|2) \oplus \mathfrak{psu}(2|2)$ symmetry. Remarkably, the symmetry completely determines the dispersion relation and also the S-matrix up to an overall scalar factor [5]. Given the scalar factor as a function of the momenta and the coupling, one can systematically study the spectrum of the system by making use of powerful techniques developed for conventional integrable models, such as the Bethe ansatz.

The determination of the scalar factor, or equivalently its principal part called the dressing phase, was one of the main outstanding problems in this field. The entire expression of the dressing phase as the strong coupling expansion was first constructed [12]

so that it satisfies the crossing symmetry [13] and includes the previously known first two terms [14–17] which reproduce the semi-classical string spectrum. Subsequently, a systematic way of its determination as the weak coupling expansion was presented [18]: The problem can be rephrased in terms of the cusp anomalous dimension [19] where the transcendentality principle [20], with some empirical rules, fully determines the dressing phase up to an overall multiplicative constant. The constant is readily singled out by comparison with either the perturbative computation [21] or the strong coupling result [12]. Ultimately the weak coupling result is identified with the strong coupling one by a sort of analytic continuation [18, 22] and is nicely expressed in a closed integral formula. We call it the BHL/BES dressing phase after the authors of the articles [12, 18]. Its properties, such as the pole structure [23] as well as the strong coupling limit [24–27], have been further studied.

Despite the success in the determination, the clear understanding of the scalar factor was still lacking. The above procedures do not explain why the scalar factor should exhibit its particular structure. It is also unsatisfactory that these procedures require some model-specific computation of the string/gauge theory. Although there are some interesting results explaining part of its structure [23, 28–30], one would desire a comprehensive explanation.

Let us recall here that in the field of integrable models, there are two well-known approaches for the computation of the S-matrices: One is called the factorized bootstrap program or the phenomenological computation [31], the other is called the direct calculation, the microscopic derivation or the Bethe ansatz technique [32–34].

The former approach is to compute the S-matrices as an inverse problem. In two-dimensional massive relativistic integrable models, two-body S-matrices of the fundamental particles satisfy the unitarity, the factorizability, and the crossing symmetry. These conditions constrain the form of the S-matrices up to the CDD ambiguity. The ambiguity can be removed by some additional requirements, such as the absence of the poles corresponding to unphysical particles.

The latter approach is to compute the S-matrices as a direct problem. For example, in the anti-ferromagnetic Heisenberg spin-chain the physical vacuum is the anti-ferromagnetic state rather than the ferromagnetic state. The anti-ferromagnetic state is realized as a nontrivial solution of the bare Bethe ansatz equations built over the ferromagnetic reference state. In other words, the physical vacuum is constructed by filling up the Dirac sea over the bare vacuum. The R-matrix describes the scattering of the magnons, which are the fundamental excitations over the bare vacuum. On the other hand, the S-matrix appears as the scattering matrix of the spinons, which are the fundamental excitations over the physical vacuum.¹

The above mentioned determination [12] of the scalar factor of the AdS/CFT S-matrix basically followed the former bootstrap program. It is natural to expect that one could

¹The R-matrix (the scattering matrix of magnons) is proportional to the S-matrix and has a trivial scalar factor. It also satisfies the unitarity, the factorizability, but does not satisfy the crossing symmetry. In the latter approach one does not assume the crossing symmetry anywhere. Instead, the S-matrix, which describes the physical scattering, becomes crossing-symmetric automatically, even though one starts from the crossing-non-invariant R-matrix.

determine the scalar factor alternatively by the latter direct computation. The idea of such nontrivial structure of the physical vacuum in the context of the AdS/CFT correspondence has been sometimes considered [35]. A concrete hint was observed in a computation of all-order anomalous dimensions [36]. The authors of [36] derived the integral equation describing the all-order anomalous dimensions of field strength operators and found that there appear integral kernels very similar to those describing the scalar factor. Such kernels are generated by the elimination of density functions of Bethe roots at nested levels. In our previous article [37], we demonstrated that a certain configuration of the Bethe roots at nested levels indeed generates the dressing phase in the all-order Bethe equations.²

In this article, we investigate in detail this microscopic formulation of the AdS/CFT S-matrix. After a brief review of the S-matrix and the all-order Bethe equations in section 2, we present in section 3 the whole configuration of the Bethe roots describing the physical vacuum. The configuration consists of a long Bethe string stretched along the imaginary axis and stacks distributed along the real axis. The former part corresponds to the configuration of a pulsating string, while the latter is analogous to the vacuum configuration of the Hubbard model in the attractive case. We determine the density distribution of the stacks. In section 4, we subsequently compute the density of stacks in the presence of fundamental excitations. Using this, we directly compute the S-matrix as the two-body scattering matrix of the fundamental excitations over the physical vacuum, to find precisely the BHL/BES dressing phase [12, 18]. Section 5 is devoted to a discussion. The derivation of the effective momentum phase of stacks is presented in appendix A.

2. S-matrix and nested Bethe ansatz equations

Let us start our discussion with an introduction of some notations. The S-matrix is most concisely expressed with the help of the following parametrizations

$$x^\pm(u) = x(u \pm \frac{i}{2}), \quad x(u) = \frac{u}{2} \left(1 + \sqrt{1 - 4g^2/u^2}\right). \quad (2.1)$$

Here u is an analogue of the rapidity parameter and

$$g = \frac{\sqrt{\lambda}}{4\pi} \quad (2.2)$$

is the normalized coupling constant. In terms of these parameters, the momentum p of a fundamental particle is expressed as

$$e^{ip} = \frac{x^+}{x^-}. \quad (2.3)$$

The scattering matrix appearing in the context of the AdS/CFT correspondence exhibits the following tensor product structure

$$\hat{S}(p_k, p_j) = S_0(p_k, p_j)^2 [\hat{R}(p_k, p_j) \otimes \hat{R}(p_k, p_j)], \quad (2.4)$$

²After the submission of our previous article [37], there appeared a similar computation in the revised version of [36]. While the mechanism of the generation of the dressing phase is essentially the same, their formulation looks conceptually different from ours. For instance, there appear Bethe roots at the nested levels twice as many kinds as ours.

where $\hat{R}(p_k, p_j)$ is the $\mathfrak{su}(2|2)$ invariant R-matrix of size 16×16 and $S_0(p_k, p_j)^2$ is the overall scalar factor. The form of the R-matrix is completely determined by the symmetry [5, 10]. There are some variations of the canonical form of the R-matrix, depending on the choice of the basis. Here we adopt the string theory basis [11] so that the R-matrix satisfies the ordinary Yang-Baxter algebra.

The overall scalar factor is conventionally expressed as

$$S_0(p_k, p_j)^2 = \frac{x_k^- - x_j^+}{x_k^+ - x_j^-} \frac{1 - g^2/x_k^+ x_j^-}{1 - g^2/x_k^- x_j^+} e^{2i\theta(u_k, u_j)}, \quad (2.5)$$

where $2\theta(u_k, u_j)$ is called the dressing phase [14]. If we regard the S-matrix (2.4) as the scattering matrix of physical particles, the dressing phase turns out to be a nontrivial function. Its form was recently determined [12, 18]. Let us call it BHL/BES dressing phase. It is expressed as

$$2\theta_{\text{phys}}(u_k, u_j) = 2ig^2 \int_{-\infty}^{\infty} dt e^{itu_k} e^{-\frac{|t|}{2}} \int_{-\infty}^{\infty} dt' e^{it'u_j} e^{-\frac{|t'|}{2}} \left(\hat{K}_d(2gt, 2gt') - \hat{K}_d(2gt', 2gt) \right), \quad (2.6)$$

where the Fourier transform is a skew combination of the dressing kernel

$$\hat{K}_d(t, t') = 8g^2 \int_0^{\infty} dt'' \hat{K}_1(t, 2gt'') \frac{t''}{e^{t''} - 1} \hat{K}_0(2gt'', t'). \quad (2.7)$$

The constituent kernels are given by

$$\hat{K}_0(t, t') = \frac{tJ_1(t)J_0(t') - t'J_0(t)J_1(t')}{t^2 - t'^2}, \quad \hat{K}_1(t, t') = \frac{t'J_1(t)J_0(t') - tJ_0(t)J_1(t')}{t^2 - t'^2}, \quad (2.8)$$

where $J_n(t)$ are Bessel functions of the first kind.

The goal of the present article is to derive this BHL/BES dressing phase in the context of the underlying bare integrable model. In other words, we describe the system starting from a bare vacuum where the scattering matrix of the fundamental excitations has the same structure as (2.4)–(2.5) but with the trivial dressing phase

$$2\theta_{\text{bare}}(u_k, u_j) = 0. \quad (2.9)$$

In the bare description, the physical S-matrix can be computed as the scattering matrix of the fundamental excitations over the Fermi surface.

Given the form of the S-matrix, one can derive a set of Bethe ansatz equations. Let us consider the system of N particles in a periodic one-dimensional box of length L . We impose integrability of the system, namely the condition that any multi-body scattering is factorized into a product of two-body scatterings described by the above S-matrix. For simplicity we consider the case of zero total momentum

$$P = \sum_{j=1}^N p_j = 0. \quad (2.10)$$

The consistency conditions for the periodicity give rise to the Yang equations

$$e^{ip_k L} = \prod_{j \neq k}^N \hat{S}(p_k, p_j). \quad (2.11)$$

These matrix equations are diagonalized with the help of the nested Bethe ansatz. Combined with the momentum condition (2.10), the nested Bethe ansatz equations can be expressed as the asymptotic all-order Bethe ansatz equations [4]

$$1 = \prod_{j=1}^{K_4} \frac{x_{4,j}^+}{x_{4,j}^-}, \quad (2.12)$$

$$1 = \prod_{j=1}^{K_2} \frac{u_{1,k} - u_{2,j} + i/2}{u_{1,k} - u_{2,j} - i/2} \prod_{j=1}^{K_4} \frac{1 - g^2/x_{1,k} x_{4,j}^+}{1 - g^2/x_{1,k} x_{4,j}^-}, \quad (2.13)$$

$$1 = \prod_{j \neq k}^{K_2} \frac{u_{2,k} - u_{2,j} - i}{u_{2,k} - u_{2,j} + i} \prod_{j=1}^{K_3} \frac{u_{2,k} - u_{3,j} + i/2}{u_{2,k} - u_{3,j} - i/2} \prod_{j=1}^{K_1} \frac{u_{2,k} - u_{1,j} + i/2}{u_{2,k} - u_{1,j} - i/2}, \quad (2.14)$$

$$1 = \prod_{j=1}^{K_2} \frac{u_{3,k} - u_{2,j} + i/2}{u_{3,k} - u_{2,j} - i/2} \prod_{j=1}^{K_4} \frac{x_{3,k} - x_{4,j}^+}{x_{3,k} - x_{4,j}^-}, \quad (2.15)$$

$$\left(\frac{x_{4,k}^+}{x_{4,k}^-} \right)^J = \prod_{j \neq k}^{K_4} \frac{u_{4,k} - u_{4,j} + i}{u_{4,k} - u_{4,j} - i} e^{2i\theta(u_{4,k}, u_{4,j})} \prod_{j=1}^{K_1} \frac{1 - g^2/x_{4,k}^- x_{1,j}}{1 - g^2/x_{4,k}^+ x_{1,j}} \prod_{j=1}^{K_3} \frac{x_{4,k}^- - x_{3,j}}{x_{4,k}^+ - x_{3,j}} \times \prod_{j=1}^{K_7} \frac{1 - g^2/x_{4,k}^- x_{7,j}}{1 - g^2/x_{4,k}^+ x_{7,j}} \prod_{j=1}^{K_5} \frac{x_{4,k}^- - x_{5,j}}{x_{4,k}^+ - x_{5,j}}, \quad (2.16)$$

$$1 = \prod_{j=1}^{K_6} \frac{u_{5,k} - u_{6,j} + i/2}{u_{5,k} - u_{6,j} - i/2} \prod_{j=1}^{K_4} \frac{x_{5,k} - x_{4,j}^+}{x_{5,k} - x_{4,j}^-}, \quad (2.17)$$

$$1 = \prod_{j \neq k}^{K_6} \frac{u_{6,k} - u_{6,j} - i}{u_{6,k} - u_{6,j} + i} \prod_{j=1}^{K_5} \frac{u_{6,k} - u_{5,j} + i/2}{u_{6,k} - u_{5,j} - i/2} \prod_{j=1}^{K_7} \frac{u_{6,k} - u_{7,j} + i/2}{u_{6,k} - u_{7,j} - i/2}, \quad (2.18)$$

$$1 = \prod_{j=1}^{K_6} \frac{u_{7,k} - u_{6,j} + i/2}{u_{7,k} - u_{6,j} - i/2} \prod_{j=1}^{K_4} \frac{1 - g^2/x_{7,k} x_{4,j}^+}{1 - g^2/x_{7,k} x_{4,j}^-}. \quad (2.19)$$

The length L and the number of particle N are interpreted as

$$L = J - K_4 + \frac{1}{2}(-K_1 + K_3 + K_5 - K_7), \quad N = K_4. \quad (2.20)$$

We refer to [5, 10, 38, 39] for the details of the derivation.

3. Bethe root configuration of the physical vacuum

In the bare description, physical states are characterized by solutions of the bare Bethe ansatz equations, that is, the simultaneous equation (2.12)–(2.19) with the trivial dressing phase (2.9). In this section we present a particular solution that should express the nontrivial physical vacuum state.

3.1 General structure

The configuration consists of the following occupation numbers of bare Bethe roots³

$$(K_1, \dots, K_7) = (2M, M, 0, 2M, 0, M, 2M). \tag{3.1}$$

For the vacuum state the configuration of Bethe roots must be symmetric with respect to the interchange of the two $\mathfrak{su}(2|2)$ sectors: distribution of roots $u_{1,k}, u_{2,k}, u_{3,k}$ is just the same as that of $u_{7,k}, u_{6,k}, u_{5,k}$, respectively. Regarding this symmetry, we mostly omit to mention the former copy of roots hereafter.

The vacuum has to be neutral with respect to the pair of $\mathfrak{su}(2|2)$ symmetries. This restricts the relative numbers of the Bethe roots to be $K_4 = K_5 + K_7 = 2K_6$. It can be understood as follows: We restrict ourselves on one of the $\mathfrak{su}(2|2)$'s. Let $[n_1; n_2]$ denote the two $\mathfrak{su}(2)$ charges, by which we mean the Dynkin indices with respect to the bosonic subalgebra $\mathfrak{su}(2) \oplus \mathfrak{su}(2) \subset \mathfrak{su}(2|2)$. A bosonic root u_4 creates a magnon with charges $[1; 0]$. Either of fermionic roots u_5 or u_7 converts the magnon charges $[1; 0]$ to $[0; 1]$. A bosonic root u_6 flips the latter $\mathfrak{su}(2)$ spin down, namely it converts $[0; 1]$ to $[0; -1]$. Either u_5 or u_7 converts $[0; -1]$ to $[-1; 0]$. In other words, u_5 and u_7 have charges $[-1; 1]$ while u_6 has charges $[0; -2]$. It then follows that a state with general excitations has charges $[K_4 - K_5 - K_7; K_5 + K_7 - 2K_6]$.

The distribution among K_5 and K_7 is not determined by the neutralness of the vacuum, since the Bethe roots u_5 and u_7 originate in the same nested level of diagonalization [5] and thus carry the same $\mathfrak{su}(2|2)$ charges. In fact, the Bethe roots $x_{7,k}$ are introduced by the relabeling $x_7 = g^2/x_5$ [4] in order to recover the seven sets of equations (2.13)–(2.19) out of five [5, 38].

Distinction of the roots u_5 and u_7 arises in connection with the $\mathfrak{su}(2, 2|4)$ one-loop Bethe equations. For a general value of u , the value of $x(u)$ has an ambiguity of the square root branch (2.1). The branch of x_5 and x_7 are chosen so that x_5, x_7 approach u_5, u_7 , respectively, in the one-loop limit $g \rightarrow 0$. In other words, we relabel x_5 and x_7 through the relation $x_7 = g^2/x_5$ so that all x_5 's and x_7 's satisfy $|x_5| > g, |x_7| > g$. The vacuum configuration has no x_5 root, hence all the roots at the nested levels decouple from u_4 's in the one-loop limit. The set of occupation numbers (3.1) are not allowed for the one-loop Bethe equations. The vacuum configuration is characteristic of the all-order Bethe equations.

When we derive the dressing phase, we send both J and M to infinity. However, for the purpose of studying the vacuum configuration, it is convenient to take the limit $J \rightarrow \infty$ first while keeping M sufficiently large but finite. We postpone taking the limit $M \rightarrow \infty$ until we discuss excited states in the next section.

In what follows we will specify the whole configuration of the bare Bethe roots.

³The occupation numbers (3.1) satisfy the condition $K_2 \leq K_1 + K_3 \leq K_4 \geq K_5 + K_7 \geq K_6$ required for the all-order Bethe ansatz equations. (This condition follows from the consistency of nested Bethe ansatz. See, e.g. [38].) On the other hand, they are outside the bound $K_1 \leq K_2 \leq K_3 \leq K_4 \geq K_5 \geq K_6 \geq K_7$ required for the one-loop Bethe ansatz equations. This means that the vacuum configuration is characteristic of all-order Bethe ansatz equations and becomes singular in the one-loop limit.

3.2 Configuration of the central Bethe roots

In our previous article [37], we discussed mostly the configuration of Bethe roots other than the u_4 roots, which is the most essential part of the derivation of the dressing phase. Here, we further specify the precise configuration of the u_4 roots of the vacuum state.

The configuration of u_4 's is extremely simple when we see it on the u -plane. For $J \rightarrow \infty$, it is given by

$$u_{4,k} = \tilde{k}i \tag{3.2}$$

in terms of a shifted index $\tilde{k} = k - M - \frac{1}{2}$, which runs over

$$\tilde{k} = -M + \frac{1}{2}, -M + \frac{3}{2}, \dots, M - \frac{3}{2}, M - \frac{1}{2}. \tag{3.3}$$

This configuration looks like nothing but a conventional Bethe string of length $2M$. In the present case, however, it is not enough to specify only the values of $u_{4,k}$'s because for each $x_{4,k}$ there is a choice of two branches of the square root (2.1). We choose them in such a way that $\text{Im } x_{4,k}^+ > 0$, $\text{Im } x_{4,k}^- < 0$ for all roots. In other words, the vacuum configuration is completely specified on the x -plane. Explicitly, it is given by

$$x_{4,k}^\pm = \frac{i}{2} \left(\tilde{k} \pm \frac{1}{2} \pm \sqrt{\left(\tilde{k} \pm \frac{1}{2} \right)^2 + 4g^2} \right). \tag{3.4}$$

Note that the distribution (3.2) on the u -plane is common to the magnon bound state [40]. For that state, however, the choice of branches is $x_{4,k}^+ = x_{4,k+1}^-$ for $k = 1, \dots, 2M - 1$ and $\text{Im } x_{4,1}^- < 0$, $\text{Im } x_{4,2M}^+ > 0$, which is different from (3.4).

Several comments are in order. First, it is very natural that the configuration is simple and, in particular, does not have any continuous modulus parameter. There is only one discrete parameter, the number of u_4 roots $2M$, which will be eventually sent to infinity.

Second, the vacuum configuration (3.2) for large M is transparent when scattered with extra u_4 roots. More precisely, when one scatters an extra root u_4 with the vacuum configuration (3.2), it gains a scattering phase against each constituent $u_{4,k}$. However, there occurs cancellation and thus the total scattering phase is

$$\prod_{j=1}^{2M} \frac{u_4 - u_{4,j} + i}{u_4 - u_{4,j} - i} = \frac{u_4 + (M + \frac{1}{2})i}{u_4 - (M + \frac{1}{2})i} \frac{u_4 + (M - \frac{1}{2})i}{u_4 - (M - \frac{1}{2})i}, \tag{3.5}$$

which becomes trivial in the large M limit. This property is common to the magnon bound states and is crucial later in the computation of the dressing phase where we in fact add extra u_4 roots to the vacuum. On the other hand, in contrast to the case of the magnon bound state, there occurs no cancellation in the parts where $x_{4,k}$'s appear explicitly in the Bethe equations. This is necessary for having a sufficient number of stack solutions; otherwise such a cancellation in (2.19) decreases the number of solutions of $u_{7,k}$ satisfying $|x_{7,k}| > g$ less than $2M$.

Third, the vacuum solution is characteristic of the all-order Bethe ansatz equations: If we take the one-loop limit $g \rightarrow 0$, the configuration (3.4) becomes singular, which is in

accord with the fact that the dressing phase vanishes in this limit. This is again in contrast to the magnon bound state, which survives for $g \rightarrow 0$ with each pair of $x_{4,k}^\pm$ approaching $u_{4,k} \pm i/2$.

Some readers might wonder whether the above Bethe string with the present branch choice really exists, though it solves the Bethe equations in the limit $J \rightarrow \infty$, and how to understand such a singular behavior in the one-loop limit. To answer this question, it would be instructive to consider the configuration temporarily in the physical Bethe equations, where we can make use of the correspondence with classical strings. The deviation due to the presence of the dressing phase is within the error of the string hypothesis and is negligible for $J \rightarrow \infty$. Let us consider the thermodynamic limit $J \rightarrow \infty$, keeping g, M proportional to J , and introduce a rescaled spectral parameter $\tilde{x} = x/g$. On the \tilde{x} -plane the imaginary roots (3.4) form two condensates $[-ib, -ib^{-1}]$, $[ib^{-1}, ib]$ with $b = M/2g + \sqrt{(M/2g)^2 + 1}$. Because the configuration is symmetric under the interchange $\tilde{x} \leftrightarrow 1/\tilde{x}$, the corresponding classical string lives in the $S^2 \times \mathbb{R}$ sector [41]. There are few candidates for the solution in the $S^2 \times \mathbb{R}$ sector with only one modulus b . We identify it as a pulsating string [44, 43, 42] (see also [45, 46]). Pulsating string is an elliptic solution and has a continuous elliptic modulus k and a discrete winding number. The winding number is read from the density of the imaginary Bethe roots on the u -plane, which is 1 in this case. Given the winding number, the elliptic modulus k is determined by b . When we send M to infinity, b also goes to infinity and k approaches 0.⁴ Thus the configuration (3.4) with large M corresponds to the rational limit of the pulsating string. It sweeps the S^2 at almost constant speed with high frequency.

An unusual feature of this configuration is that the condensates run across the unit circle on the \tilde{x} -plane. Such a solution is precisely an exception to the general correspondence between classical strings and solutions of the one-loop Bethe equations [42, 47, 48].

Pulsating string solution has zero angular momentum in the S^2 . This is akin to the neutrality of the anti-ferromagnetic state in a spin-chain.

3.3 Formation of stacks

In our vacuum configuration, the Bethe roots u_6 and u_7 form stacks [37]

$$u_{7,2k-1} = u_{6,k} + \frac{i}{2}, \quad u_{7,2k} = u_{6,k} - \frac{i}{2}, \quad \text{for } k = 1, \dots, M. \quad (3.6)$$

Without knowing the bare Hamiltonian, one cannot verify that this configuration really corresponds to the ground state. However, most likely it does, by analogy with the Hubbard model. The Bethe equations (2.18)–(2.19) resemble very much the Lieb-Wu equations for the one-dimensional Hubbard model. The vacuum of the Hubbard model was well studied [49, 50]. In the attractive case, the vacuum consists of precisely this kind of stacks [50], namely a kind of k - Λ strings [51]. Note that this kind of stack also appears in the description of the field strength operators $\text{Tr } \mathcal{F}^L$ [36].

⁴Although one can take k arbitrarily small, it cannot be strictly zero as far as the winding number is nonzero. The strictly rational case $k = 0$ corresponds to zero winding number, which looks no longer a pulsating string but rather a point-like string.

Multiplying Bethe equations for $u_{7,2k-1}$ and for $u_{7,2k}$ together, one obtains the following set of Bethe equations

$$1 = \prod_{j=1}^{2M} \frac{1 - g^2/x_{6,k}^+ x_{4,j}^+}{1 - g^2/x_{6,k}^+ x_{4,j}^-} \frac{1 - g^2/x_{6,k}^- x_{4,j}^+}{1 - g^2/x_{6,k}^- x_{4,j}^-} \prod_{j \neq k}^M \frac{u_{6,k} - u_{6,j} + i}{u_{6,k} - u_{6,j} - i}. \quad (3.7)$$

They can be viewed as effective Bethe equations for $u_{6,k}$ denoting the centers of stacks.

3.4 Distribution of stacks

Given the configuration of $x_{4,k}^\pm$, (3.7) can be viewed as Bethe equations for a single kind of Bethe roots with a regular form of self-interaction:

$$e^{i\Phi(u_{6,k})} = \prod_{j \neq k}^M \frac{u_{6,k} - u_{6,j} + i}{u_{6,k} - u_{6,j} - i}, \quad (3.8)$$

where

$$\Phi(u_{6,k}) = \frac{1}{i} \sum_{j=1}^{2M} \ln \frac{1 - g^2/x_{6,k}^+ x_{4,j}^-}{1 - g^2/x_{6,k}^+ x_{4,j}^+} \frac{1 - g^2/x_{6,k}^- x_{4,j}^-}{1 - g^2/x_{6,k}^- x_{4,j}^+} \quad (3.9)$$

is regarded as the virtual momentum phase. For sufficiently large M , one can evaluate this phase function by approximating sum by integral. We relegate the detail of calculation to appendix A. If we take M and u sufficiently large compared to the coupling constant g , the phase function approaches a reasonably simple form

$$\Phi(u) = 2M \left[2 \arctan \frac{u}{M} + \frac{u}{M} \ln \left(1 + \frac{M^2}{u^2} \right) \right]. \quad (3.10)$$

An important property of the function $\Phi(u)$ is that it is a monotonically increasing function. This is clear from the form of its derivative

$$\Phi'(u) = 2 \ln \left(1 + \frac{M^2}{u^2} \right). \quad (3.11)$$

The Bethe equations (3.8) are thus analogous to those of the one-dimensional Bose gas with repulsive δ -function interaction or those of the $\mathfrak{sl}(2, \mathbb{R})$ spin-chain.

The Bethe equations (3.8) can be written in the logarithmic form

$$2\pi n_k = \Phi(u_k) + 2 \sum_{j \neq k}^M \arctan(u_k - u_j). \quad (3.12)$$

We often abbreviate $u_{6,k}$ as u_k hereafter. The mode number n_k associated with the root u_k takes integer/half-integer value, depending on M is odd/even, respectively. Since the r.h.s. is monotonically increasing as a function of u_k , it follows that $n_k > n_j$ for $u_k > u_j$. For the vacuum configuration, we consider consecutive set of mode numbers. One can always relabel the u_k roots so that $u_k > u_j$ for $k > j$. The mode numbers for the vacuum configuration are then given by

$$n_k = -\frac{M-1}{2}, -\frac{M-3}{2}, \dots, \frac{M-3}{2}, \frac{M-1}{2} \quad \text{for } k = 1, \dots, M. \quad (3.13)$$

In contrast to the case of the anti-ferromagnetic vacuum of the Heisenberg chain, the present configuration does not correspond to the maximal filling over the real axis. In other words, the support of the distribution of u_k 's is a finite interval. One can see this as follows: If the real axis were occupied by u_1, \dots, u_M , an extra real root with mode number $(M+1)/2$ would have to sit at $u = \infty$ [33]. However, for $u_k = \infty$ the r.h.s. of (3.12) would take $(3M-1)\pi$ and thus $u_j < \infty$ for $2\pi n_j < (3M-1)\pi$, which is contradictory to the last argument.

We are interested in the distribution of u_k 's in the large M limit. From the form of the potential (3.10), we see that the characteristic length of the distribution of u_k 's is of order M . Regarding this, let us expand the summand of the interaction term in (3.12) as

$$\arctan(u_k - u_j) = \frac{\pi}{2} \text{sign}(u_k - u_j) - \frac{1}{u_k - u_j} + \mathcal{O}\left(\frac{1}{(u_k - u_j)^3}\right) \quad (3.14)$$

and evaluate the summation term by term. One finds that the sum of the first term precisely gives rise to the mode number

$$\frac{1}{2} \sum_{j \neq k}^M \text{sign}(u_k - u_j) = n_k, \quad (3.15)$$

while the sum of the lower order terms after the second one becomes negligible. The Bethe equations (3.12) then reduce to

$$\Phi(u_k) = 2 \sum_{j \neq k}^M \frac{1}{u_k - u_j}. \quad (3.16)$$

In the continuous limit, one can replace the sum by the principal-value integral and obtains

$$\Phi(u) = 2 \int_{-B}^B \frac{\rho(v) dv}{u - v}, \quad (3.17)$$

where we introduce the density function as

$$\rho(u) = \sum_{j=1}^M \delta(u - u_j). \quad (3.18)$$

As we mentioned above the density has a finite support, which is denoted by $[-B, B]$.

The integral equation (3.17) can be solved by the inverse Hilbert transformation

$$\rho(u) = \frac{1}{2\pi^2} \int_{-B}^B \sqrt{\frac{B^2 - u^2}{B^2 - v^2}} \frac{\Phi(v) dv}{v - u}. \quad (3.19)$$

The endpoints $\pm B$ are determined by the normalization condition

$$\int_{-B}^B \rho(u) du = M. \quad (3.20)$$

By evaluating these integral expressions, one obtains the distribution of the stacks.

For our later purpose, let us estimate the order of B with respect to M . It is convenient to rewrite the above equations in terms of the rescaled variables

$$u = M\tilde{u}, \quad B = M\tilde{B}. \quad (3.21)$$

We also introduce the normalized functions

$$\tilde{\Phi}(\tilde{u}) = 2 \left[2 \arctan \tilde{u} + \tilde{u} \ln \left(1 + \frac{1}{\tilde{u}^2} \right) \right], \quad (3.22)$$

$$\tilde{\rho}(\tilde{u}) = \frac{1}{M^2} \sum_{j=1}^M \delta(\tilde{u} - \tilde{u}_j), \quad (3.23)$$

which are related to the original functions by

$$\Phi(u) = M\tilde{\Phi}(\tilde{u}), \quad \rho(u) = M\tilde{\rho}(\tilde{u}). \quad (3.24)$$

In terms of these rescaled quantities, the integral equation (3.17) gives rise to

$$\tilde{\Phi}(\tilde{u}) = 2 \int_{-\tilde{B}}^{\tilde{B}} \frac{\tilde{\rho}(\tilde{v}) d\tilde{v}}{\tilde{u} - \tilde{v}}. \quad (3.25)$$

This equation is formally the same as (3.17), thus the solution is given by (3.19) with all quantities replaced by the rescaled ones. Note that M -dependence now enters only through the endpoint value \tilde{B} , which is determined by the normalization condition

$$\int_{-\tilde{B}}^{\tilde{B}} \tilde{\rho}(\tilde{u}) d\tilde{u} = \frac{1}{M}. \quad (3.26)$$

It is clear that \tilde{B} becomes small as one sends M large. This means that for large M , $\tilde{\rho}(\tilde{u})$ is determined by the form of $\tilde{\Phi}(\tilde{u})$ only at small \tilde{u} . Except for the very vicinity of the origin, $\tilde{\Phi}(\tilde{u})$ at small \tilde{u} roughly behaves as a linear function

$$\tilde{\Phi}(\tilde{u}) \sim 4\tilde{u} \ln \frac{1}{\tilde{B}_0}, \quad (3.27)$$

where $\tilde{B}_0 \sim \tilde{B}$ is a typical scale. With this approximation one can analytically solve the integral equation and obtains

$$\tilde{\rho}(\tilde{u}) \sim \frac{2}{\pi} \left(\ln \frac{1}{\tilde{B}_0} \right) \sqrt{\tilde{B}^2 - \tilde{u}^2}. \quad (3.28)$$

The normalization condition (3.26) now reads

$$\tilde{B}^2 \ln \frac{1}{\tilde{B}_0} \sim \frac{1}{M}. \quad (3.29)$$

Ignoring the correction coming from the logarithm, one finds that \tilde{B} roughly scales with $M^{-1/2}$. Thus the original B roughly scales with $M^{1/2}$.

4. Excited states and computation of the scattering phase

In this section we consider excited states and compute the two-body S-matrix of fundamental excitations. By making use of the underlying symmetry, the S-matrix can be written most generally in the form of the spectral decomposition. As the symmetry fixes the form of the projectors, it is enough to determine the eigenvalues in front of the projectors. For example, the $\mathfrak{su}(2)$ Zamolodchikovs' S-matrix is constructed by computing the scattering phases with respect to the triplet and the singlet. In the present case, the centrally extended $\mathfrak{su}(2|2)$ algebra possesses a peculiar feature that the tensor product of a pair of 4-dimensional atypical representations is irreducible [10]. Therefore it is enough to compute only one scattering phase of a pair of fundamental excitations in a representative state.⁵ This allows us to restrict our consideration of excited states to those with only u_4 roots added.

4.1 Fundamental excitations

Let us consider excited states by adding extra u_4 roots to the vacuum configuration. We let $w_{4,k}$ denote the extra roots and N_4 be their total number. The occupation numbers read

$$(K_1, \dots, K_7) = (2M, M, 0, 2M + N_4, 0, M, 2M). \quad (4.1)$$

We keep the structure of the other roots unchanged, namely the other $2M$ u_4 's constitute the Bethe string with the branch choice (3.4), and the u_6 's and the u_7 's form M stacks with consecutive mode numbers. In this subsection let us determine the deviation of the density of stacks for fixed w_4 's.

The effective Bethe equations for the centers of stacks read

$$2\pi n_k = \Phi_{\text{ex}}(u_k) + 2 \sum_{j \neq k}^M \arctan(u_k - u_j), \quad (4.2)$$

with mode numbers (3.13). The only difference from (3.12) is the momentum phase

$$\Phi_{\text{ex}}(u) = \Phi(u) + \varphi(u), \quad (4.3)$$

where the modification part is given by

$$\varphi(u_{6,k}) = \frac{1}{i} \sum_{j=1}^{N_4} \ln \frac{1 - g^2/x_{6,k}^+ y_{4,j}^-}{1 - g^2/x_{6,k}^+ y_{4,j}^+} \frac{1 - g^2/x_{6,k}^- y_{4,j}^-}{1 - g^2/x_{6,k}^- y_{4,j}^+} \quad (4.4)$$

with $y_{4,j}^\pm = x^\pm(w_{4,j})$. By taking the derivative with respect to u_k , (4.2) gives rise to

$$2\pi \rho_{\text{ex}}(u) = \Phi'_{\text{ex}}(u) + 2 \int_{-B_{\text{ex}}}^{B_{\text{ex}}} \frac{\rho_{\text{ex}}(v) dv}{(u-v)^2 + 1}, \quad (4.5)$$

⁵The computation presumes that the physical vacuum is a singlet. In the last section we constructed the vacuum as a neutral state under the pair of $\mathfrak{su}(2|2)$ symmetries. For the centrally extended algebra, however, the state has to be neutral with respect to the central charges as well. We define the action of the central charges in our bare integrable model so that the physical vacuum has zero central charges, by shifting one of the central charges of the reference vacuum.

where $\rho_{\text{ex}}(u)$ is the density function of the centers of stacks under the modified momentum phase. Apparently, one could derive the same integral equation for the vacuum density $\rho(u)$ with the vacuum phase $\Phi(u)$. Subtracting it from (4.5), one obtains an integral equation for the density deviation $\sigma(u) = \rho_{\text{ex}}(u) - \rho(u)$, as follows

$$2\pi\sigma(u) = \varphi'(u) + 2 \int_{-B}^B \frac{\sigma(v)dv}{(u-v)^2 + 1} + 2 \left(\int_{-B_{\text{ex}}}^{-B} + \int_B^{B_{\text{ex}}} \right) \frac{\rho_{\text{ex}}(v)dv}{(u-v)^2 + 1}. \quad (4.6)$$

We consider the large M limit keeping the other parameters $w_{4,k}, N_4$ fixed. In this case the second integral is negligible: The fluctuation of the overall shape of $\Phi(u)$ should be suppressed at most within the change of M to $M + \Delta M$ with $\Delta M = \mathcal{O}(1)$. Then the deviation of $B \sim M^{1/2}$ is at most $\Delta B \sim (M + \Delta M)^{1/2} - M^{1/2} \sim M^{-1/2}\Delta M$. On the other hand, we saw in the last section that $\rho(u) \sim \ln(M/B_0)\sqrt{B^2 - u^2}$. Then the second integral is suppressed by $\ln(M/B_0)B^{1/2}(\Delta B)^{3/2} \sim M^{-1/2}\ln M$, which vanishes when M is sent to infinity. After all, in the limit $M \rightarrow \infty$ we obtain

$$2\pi\sigma(u) = \varphi'(u) + 2 \int_{-\infty}^{\infty} \frac{\sigma(v)dv}{(u-v)^2 + 1}. \quad (4.7)$$

This integral equation is solved in the Fourier space [19, 36]. Using the techniques in the appendix D of [19], one can derive the following formulas⁶

$$\ln(1 - g^2/x^\pm(u)x^\pm(u')) = 2g^2 \int_0^\infty dt e^{\pm iut} e^{-t/2} \int_0^\infty dt' e^{\pm iu't'} e^{-t'/2} \hat{H}_m(2gt, 2gt'), \quad (4.8)$$

$$\ln(1 - g^2/x^\pm(u)x^\mp(u')) = -2g^2 \int_0^\infty dt e^{\pm iut} e^{-t/2} \int_0^\infty dt' e^{\mp iu't'} e^{-t'/2} \hat{K}_m(2gt, 2gt'), \quad (4.9)$$

where the integral kernels are expressed in terms of Bessel functions $J_n(t)$ by

$$\hat{H}_m(t, t') = \frac{J_1(t)J_0(t') + J_0(t)J_1(t')}{t + t'}, \quad \hat{K}_m(t, t') = \frac{J_1(t)J_0(t') - J_0(t)J_1(t')}{t - t'}. \quad (4.10)$$

With these formulas, one immediately obtains the solution in the Fourier space

$$\sigma(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt e^{itu} \hat{\sigma}(t), \quad (4.11)$$

where

$$\hat{\sigma}(\pm t) = -\frac{g^2 t}{\sinh \frac{t}{2}} \sum_{j=1}^{N_4} \int_0^\infty dt' e^{-t'/2} \left(e^{\pm i t' w_{4,j}} \hat{H}_m(2gt, 2gt') + e^{\mp i t' w_{4,j}} \hat{K}_m(2gt, 2gt') \right), \quad (4.12)$$

for $t > 0$.

⁶We assume $u, u' \in \mathbb{R}$. The branch of logarithm should be chosen appropriately.

4.2 The dressing phase

We are now in a position to compute the dressing phase. The vacuum configuration in the bare description corresponds to the empty state in the physical description. The physical fundamental excitations are described by adding extra roots $w_{4,k}$ in the bare description. The bare configuration we studied in the last subsection corresponds to the system of N_4 excitations.

In the physical description, the scattering phase of the two fundamental excitations is simply given by

$$\phi_{12} = \frac{1}{i} \ln \frac{w_{4,1} - w_{4,2} + i}{w_{4,1} - w_{4,2} - i} + 2\theta_{\text{phys}}(w_{4,1}, w_{4,2}). \quad (4.13)$$

In the bare description, the same scattering phase is expressed by the difference of two phases

$$\phi_{12} = \delta_{12}(w_{4,1}) - \delta_1(w_{4,1}). \quad (4.14)$$

Here δ_{12} is the total phase which the first excitation gains when moving around the chain in the presence of the second excitation. δ_1 is measured in the same way but in the absence of the second excitation [32].

The total phase is the phase of the transfer matrix eigenvalue and thus can be read from the r.h.s. of the central Bethe equations (2.16). By substituting (3.6) (and the corresponding relations for $u_{1,k}, u_{2,k}$), they read

$$\begin{aligned} \left(\frac{y_{4,k}^+}{y_{4,k}^-} \right)^J &= \prod_{j \neq k}^{N_4} \frac{w_{4,k} - w_{4,j} + i}{w_{4,k} - w_{4,j} - i} \prod_{j=1}^{2M} \frac{w_{4,k} - u_{4,j} + i}{w_{4,k} - u_{4,j} - i} \times \\ &\times \prod_{j=1}^M \frac{1 - g^2/y_{4,k}^- x_{2,j}^+}{1 - g^2/y_{4,k}^+ x_{2,j}^+} \prod_{j=1}^M \frac{1 - g^2/y_{4,k}^- x_{2,j}^-}{1 - g^2/y_{4,k}^+ x_{2,j}^-} \times \\ &\times \prod_{j=1}^M \frac{1 - g^2/y_{4,k}^- x_{6,j}^+}{1 - g^2/y_{4,k}^+ x_{6,j}^+} \prod_{j=1}^M \frac{1 - g^2/y_{4,k}^- x_{6,j}^-}{1 - g^2/y_{4,k}^+ x_{6,j}^-}. \end{aligned} \quad (4.15)$$

The total phase is then expressed as

$$\begin{aligned} \delta_{1\dots N_4}(w_{4,k}) &= \frac{1}{i} \sum_{j \neq k}^{N_4} \ln \frac{w_{4,k} - w_{4,j} + i}{w_{4,k} - w_{4,j} - i} + \frac{1}{i} \sum_{j=1}^{2M} \ln \frac{w_{4,k} - u_{4,j} + i}{w_{4,k} - u_{4,j} - i} + \\ &+ \frac{2}{i} \int_{-\infty}^{\infty} \ln \left[\frac{1 - g^2/y_{4,1}^- x^+(u)}{1 - g^2/y_{4,1}^+ x^+(u)} \frac{1 - g^2/y_{4,1}^- x^-(u)}{1 - g^2/y_{4,1}^+ x^-(u)} \right] \rho_{\text{ex}}(u) du. \end{aligned} \quad (4.16)$$

Therefore (4.14) gives rise to

$$\begin{aligned} \phi_{12} &= \frac{1}{i} \ln \frac{w_{4,1} - w_{4,2} + i}{w_{4,1} - w_{4,2} - i} + \\ &+ \frac{2}{i} \int_{-\infty}^{\infty} \ln \left[\frac{1 - g^2/y_{4,1}^- x^+(u)}{1 - g^2/y_{4,1}^+ x^+(u)} \frac{1 - g^2/y_{4,1}^- x^-(u)}{1 - g^2/y_{4,1}^+ x^-(u)} \right] (\sigma_{12}(u) - \sigma_1(u)) du, \end{aligned} \quad (4.17)$$

where $\sigma_{12}(u), \sigma_1(u)$ are given by (4.11)–(4.12) with $N_4 = 2, 1$, respectively. The first term is the bare scattering phase which the first excitation directly feels against the second one. The second term comes from the scattering of the first excitation against the stacks whose density deviation $\sigma_{12} - \sigma_1$ encodes the back reaction from the second excitation. From the comparison with (4.13), we see that the second term plays precisely the role of the dressing phase. By using the formulas (4.8)–(4.9) again, the second term can be expressed concisely in the Fourier space

$$2\theta_{\text{phys}}(w_{4,1}, w_{4,2}) = 2g^2 \int_{-\infty}^{\infty} dt e^{itw_{4,1}} e^{-\frac{|t|}{2}} \int_{-\infty}^{\infty} dt' e^{it'w_{4,2}} e^{-\frac{|t'|}{2}} \left(\hat{\mathcal{K}}(2gt, 2gt') - \hat{\mathcal{K}}(2gt', 2gt) \right), \quad (4.18)$$

where

$$\hat{\mathcal{K}}(2gt, 2gt') = 4g^2 \int_0^{\infty} dt'' \hat{K}_m(2gt, 2gt'') \frac{it''}{e^{t''} - 1} \hat{H}_m(2gt'', 2gt'). \quad (4.19)$$

This precisely agrees with the BHL/BES dressing phase (2.6).

5. Discussion

We have computed the two-body scattering phase of the fundamental excitations over the physical vacuum, which precisely agrees with the BHL/BES dressing phase. By taking account of the centrally extended $\mathfrak{psu}(2|2) \oplus \mathfrak{psu}(2|2)$ symmetry, this suffices to determine the whole 256×256 components of the S-matrix. From this S-matrix, one can construct the Yang equations and derive the complete set of physical Bethe ansatz equations, as explained in section 2. Altogether, our formulation proposes a derivation of the asymptotic all-order Bethe ansatz equations with the BHL/BES dressing phase, purely based on the symmetry and the integrability.

In the last section we have considered particular excited states that consist of only N_4 excitations. For these states the correspondence between the bare description and the physical one may be trivial in the sense that each extra bare root represents a physical root. However, the correspondence is not so simple in general: When the occupation numbers of the physical roots are still equal to that of the extra bare roots, the values of the roots could differ. In general, addition of extra Bethe roots at nested levels partly breaks the structure of the stacks. A single physical root sometimes corresponds to a complex of bare roots and holes. It would be interesting to clarify the correspondence.

For the moment we do not know whether a Yang-Mills operator of finite length can be directly realized in the bare description. A possibility is that a physical operator of length L_{ex} could be expressed by a state in the chain of length $L + L_{\text{ex}}$ while the vacuum is defined in the chain of length L . Of course both L and $L + L_{\text{ex}}$ have to be sent to infinity, but still the difference would make sense.

We have determined our vacuum configuration as the simplest consistent solution that generates the BHL/BES dressing phase. However, ultimately we wish to derive it as the ground state of a certain Hamiltonian. The Hamiltonian has to be expressed in a form compatible with the bare description, preferably in terms of the $\mathfrak{su}(2|2)$ R-matrix. It may be derived from the gauge-fixed light-cone Hamiltonian for the Green-Schwarz

superstring theory in $AdS_5 \times S^5$ [6], which exhibits the invariance under the centrally extended $\mathfrak{psu}(2|2) \oplus \mathfrak{psu}(2|2)$ symmetry when the worldsheet is decompactified [7].

Our microscopic formulation will be of fundamental use in various directions under the latest investigation, for example, the boundary S-matrix [52], the wrapping interactions [53, 54] and the Baxter equations [55, 56]. We hope to report the progress in these topics elsewhere.

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A. Effective momentum phase

The centers of stacks $u_{6,k}$ obey the following effective Bethe equations

$$e^{i\Phi(u_{6,k})} = \prod_{j \neq k}^M \frac{u_{6,k} - u_{6,j} + i}{u_{6,k} - u_{6,j} - i}, \quad (\text{A.1})$$

where

$$\Phi(u_{6,k}) = \Phi^+(u_{6,k}) + \Phi^-(u_{6,k}), \quad (\text{A.2})$$

$$\Phi^\pm(u_{6,k}) = \frac{1}{i} \sum_{j=1}^{2M} \ln \frac{1 - g^2/x_{6,k}^\pm x_{4,j}^\mp}{1 - g^2/x_{6,k}^\pm x_{4,j}^\pm}. \quad (\text{A.3})$$

Note that

$$g^2/x_{4,j}^\pm = \frac{i}{2} \left(\tilde{j} \pm \frac{1}{2} \mp \sqrt{\left(\tilde{j} \pm \frac{1}{2}\right)^2 + 4g^2} \right), \quad (\text{A.4})$$

where the index \tilde{j} is defined as

$$\begin{aligned} \tilde{j} &= j - M - \frac{1}{2} \\ &= -M + \frac{1}{2}, -M + \frac{3}{2}, \dots, M - \frac{3}{2}, M - \frac{1}{2}. \end{aligned} \quad (\text{A.5})$$

Let us evaluate the function $\Phi^+(u)$ in the large M limit:

$$\Phi^+(u) = \frac{1}{i} \sum_{j=1}^{2M} \ln \frac{1 - g^2/x_{4,j}^- x^+(u)}{1 - g^2/x_{4,j}^+ x^+(u)}$$

$$\begin{aligned}
 &= \frac{1}{i} \sum_{\tilde{j}=-M+1/2}^{M-1/2} \ln \frac{\sqrt{(\tilde{j}-\frac{1}{2})^2 + 4g^2} + (\tilde{j}-\frac{1}{2}) + 2ix^+(u)}{\sqrt{(\tilde{j}+\frac{1}{2})^2 + 4g^2} - (\tilde{j}+\frac{1}{2}) - 2ix^+(u)} \\
 &= \frac{1}{i} \sum_{\tilde{j}=-M+1/2}^{M-1/2} \ln \frac{\sqrt{t_j^2 + a^2} - t_j + b}{\sqrt{t_j^2 + a^2} - t_j - b}, \tag{A.6}
 \end{aligned}$$

where

$$t_j = \frac{\tilde{j} + \frac{1}{2}}{M}, \quad a = \frac{2g}{M}, \quad b = \frac{2ix^+(u)}{M}. \tag{A.7}$$

In the large M limit, one can approximate the sum in (A.6) by the integral

$$\Phi^+(u) = \frac{M}{i} \int_{-1}^1 dt \ln \frac{\sqrt{t^2 + a^2} - t + b}{\sqrt{t^2 + a^2} - t - b}. \tag{A.8}$$

This integral can be performed by the change of variable $s = \sqrt{t^2 + a^2} - t$. In fact,

$$\begin{aligned}
 &\int_{-1}^1 dt \ln(\sqrt{t^2 + a^2} - t + b) = \\
 &= \int_{\sqrt{1+a^2}+1}^{\sqrt{1+a^2}-1} ds \left(-\frac{s^2 + a^2}{2s^2} \right) \ln(s + b) \\
 &= \frac{1}{2} \left[\left(\frac{a^2}{s} - s \right) \ln(s + b) - b \ln(s + b) + \frac{a^2}{b} \ln \left(1 + \frac{b}{s} \right) + s + b \right]_{s=\sqrt{1+a^2}+1}^{s=\sqrt{1+a^2}-1}. \tag{A.9}
 \end{aligned}$$

Using this, one obtains

$$\begin{aligned}
 \Phi^+(u) &= \frac{M}{2} \left[2i \ln \frac{(\lambda + i\sqrt{1+a^2})^2 + 1}{(\lambda - i\sqrt{1+a^2})^2 + 1} + \right. \\
 &\quad \left. + \lambda \ln \frac{\lambda^2 + (\sqrt{1+a^2} + 1)^2}{\lambda^2 + (\sqrt{1+a^2} - 1)^2} - \frac{a^2}{\lambda} \ln \frac{a^4/\lambda^2 + (\sqrt{1+a^2} + 1)^2}{a^4/\lambda^2 + (\sqrt{1+a^2} - 1)^2} \right], \tag{A.10}
 \end{aligned}$$

where

$$\lambda = -ib = \frac{2x^+(u)}{M}, \quad a = \frac{2g}{M}. \tag{A.11}$$

$\Phi^-(u)$ takes the same form with $\lambda = 2x^-(u)/M$.

Let us consider the case where the coupling constant g is finite. As we take M and u sufficiently large, we see that

$$\lambda \approx \frac{2u}{M}, \quad a \approx 0. \tag{A.12}$$

As a result, the phase function reduces to a reasonably simple form

$$\Phi(u) = 2M \left[2 \arctan \frac{u}{M} + \frac{u}{M} \ln \left(1 + \frac{M^2}{u^2} \right) \right] - 2\pi M. \tag{A.13}$$

In the above computation we implicitly chose the branch of logarithm so that $\Phi(u) = 0$ at $u = +\infty$. In the main text we drop the constant $-2\pi M$, which corresponds to the choice of the branch where $\Phi(u) = 0$ at $u = 0$.

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